

Exact Parametrization of Majorana Neutrino Mass Matrix with Large Mixing

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Abstract

Under the assumption that the neutrinos are Majorana particles we study how the lepton mass matrices can be transformed into the simple form which has the same physical quantities by removing redundant parameters. We propose the exact parametrization of the lepton mass matrices which reflects the small $\nu_e - \nu_\mu$ mixing and the large $\nu_\mu - \nu_\tau$ mixing. The relations between the twelve parameters and the physical quantities are shown. Furthermore we calculate the MNS matrix by applying the assumptions used in the quark sector. Finally we also check the validity of these assumptions from the experimental values.

1 Introduction

The finite mass of neutrinos and the mixing among different flavor neutrinos have been suggested by various neutrino oscillation experiments. One of the major experiments is the solar neutrino experiments, which indicate the oscillation between ν_e and other neutrinos. Another one is the atmospheric neutrino experiments, which indicate the oscillations between ν_μ and other neutrinos.

In solar neutrino experiments three possible solutions are proposed: small or large mixing MSW solution [1] and vacuum oscillation solution with large mixing [2]. Especially the small mixing MSW solution [3]

$$\sin^2 2\theta_{ex} \sim 10^{-2}, \Delta m_{\text{solar}}^2 \sim 10^{-5} \text{ eV}^2, \quad (1)$$

has been thought as the strong candidate compared with others for the solar neutrino problem. Furthermore $\nu_e \rightarrow \nu_\mu$ oscillations are the most likely channel in (1) although other channels $\nu_e \rightarrow \nu_\tau$ and $\nu_e \rightarrow \nu_s$ are not be excluded.

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Another experiments are the atmospheric neutrino experiments. In particular, the recent report by Super-Kamiokande [4] concerned with a zenith-angle-dependent deficit of ν_μ suggests the strong evidence for neutrino oscillation with large mixing,

$$\sin^2 2\theta_{\mu x} \sim 1, \Delta m_{\text{atm}}^2 \sim 10^{-3} \text{eV}^2. \quad (2)$$

Within the three neutrino picture, the atmospheric neutrino problem can be explained by $\nu_\mu \rightarrow \nu_\tau$ because $\nu_\mu \rightarrow \nu_e$ is excluded by the CHOOZ experiment [5]. In the near future it is expected that the data obtained by Super-Kamiokande, SNO, K2K and so on reveal the structure of the lepton sector more clearly.

At the moment, we recognize that the lepton sector is largely different from the quark sector in following two points. The first one is the large $\nu_\mu - \nu_\tau$ mixing in contrast to quark sector. The second one is that the neutrino masses are extremely small compared with the quark and the charged lepton masses.

These differences seem to be explained by more fundamental theory. Actually, the smallness of neutrino masses can be naturally understood as the inverse of large Majorana neutrino masses by using seesaw mechanism [6]. We can also consider that the large mixing is not originated from the charged lepton but from the Majorana neutrino. Because it is expected that the charged lepton mass matrix has the same structure as the quark one in the grand unified theory (GUT). In addition, it is expected that the contribution of the charged lepton to mixing angle is small at weak scale as well as GUT scale since the renormalization effects are small in general.

As one of the mass matrices which leads the small mixing, Fritzsch-type [7] mass matrix is proposed in the quark sector. It is given by imposing hermicity to the nearest-neighbor interactions (NNI) form, which has the components $M_{11} = M_{13} = M_{31} = M_{22} = 0$. One of approaches to explore the symmetry of more fundamental theory is to consider the simple mass matrices like this. In general Branco, Lavoura and Mota [8] proved that one can always choose the NNI form as both up and down quark mass matrices. This means that arbitrary mass matrices can be transformed into the NNI form which has the same physical quantities.

In this letter inspired by these works, we explore the simple form of Majorana neutrino mass matrix, M_ν , which only contributes to the $\nu_\mu - \nu_\tau$ mixing when we choose the NNI form as the charged lepton mass matrix M_l . As a result, we prove that one can always choose the following simple forms as Majorana neutrino mass matrix,

$$M_\nu \propto \begin{pmatrix} a_\nu & 0 & 0 \\ 0 & b_\nu & d_\nu \\ 0 & d_\nu & c_\nu \end{pmatrix}, \quad (3)$$

where a_ν, b_ν, c_ν and d_ν are complex values and are represented by five independent parameters, and charged lepton mass matrix,

$$M_l \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i\theta_2) & 0 \\ 0 & 0 & \exp(i\theta_3) \end{pmatrix} \times \begin{pmatrix} 0 & a & 0 \\ c & 0 & b \\ 0 & d & e \end{pmatrix}, \quad (4)$$

where a, b, c, d and e are all real values. Twelve parameters are contained in M_l and M_ν and just correspond to the physical quantities: three charged lepton masses, three neutrino masses, three mixing angles and three phases. Note that arbitrary M_ν and M_l can be transformed into the simple forms (3) and (4) which have the same physical quantities.

These (3) and (4) are the exact parametrization of the lepton mass matrices. In this parametrization, there is no contributions to $\nu_e - \nu_\mu$ mixing from M_ν but only from M_l . On the other hand, $\nu_\mu - \nu_\tau$ mixing is contributed from both M_l and M_ν . The large mixing can be generated only by M_ν because the contribution from M_l is small. Thus, this parametrization is suitable for the physics with the small mixing MSW solution for solar neutrino problem and the large mixing solution for atmospheric neutrino problem.

In the following section we give the proof how to transform into (3) and (4). Next we start from (3) and (4), and calculate the Maki-Nakagawa-Sakata (MNS) matrix [9], which represents lepton mixing matrix, under the assumptions used in the quark sector. Finally we check the validity of these assumptions from the experimental values.

2 Simple Form of Lepton Mass Matrices

In this section, starting from the arbitrary Majorana neutrino mass matrix M_ν and the charged lepton Dirac mass matrix M_l , we transform the neutrino mass matrix into the simple form (3) and the charged lepton mass matrix into the NNI form (4) by using the following transformation

$$M'_\nu = U^T M_\nu U, \quad (5)$$

$$M'_l = U^\dagger M_l V. \quad (6)$$

The physical quantities calculated by (M_ν, M_l) are not changed through the above transformation to (M'_ν, M'_l) . In general M_ν is a 3×3 complex symmetric matrix, which has 12 parameters and M_l is a 3×3 complex matrix, which has 18 parameters. Taking advantage of freedom contained in U and V , the number of the parameters in M_l and M_ν is reduced from 30 to 12. The remaining 12 parameters in M_l and M_ν just correspond to the physical quantities.

At first let us show how to transform M_l into NNI form, which has the components $(M'_l)_{11} = (M'_l)_{13} = (M'_l)_{31} = (M'_l)_{22} = 0$, based on the work done by Branco *et al.* [8]. Three conditions $(M'_l)_{11} = (M'_l)_{13} = (M'_l)_{31} = 0$ are satisfied by choosing the unitary matrix $V = (V_{i1}, V_{i2}, V_{i3})$ in (6) as

$$V_{i1} = N_1 \epsilon_{ijk} \mathcal{M}_{1j} \mathcal{M}_{3k}, \quad (7)$$

$$V_{i3} = N_3 (\mathcal{M}_{1i}^* (H'_l)_{13} - \mathcal{M}_{3i}^* (H'_l)_{11}), \quad (8)$$

where the N_1 and N_3 are normalization factors and \mathcal{M} , H'_l are defined by

$$\mathcal{M} \equiv U^\dagger M_l, \quad (9)$$

$$H'_l \equiv U^\dagger H_l U = U^\dagger M_l M_l^\dagger U = \mathcal{M} \mathcal{M}^\dagger. \quad (10)$$

Note that U is an arbitrary unitary matrix in (9) and (10). The vector V_{i2} is determined by the orthogonality conditions with the vectors V_{i1} and V_{i3} as

$$V_{i2} = N_2 \mathcal{M}_{1i}^*, \quad (11)$$

where the N_2 is an also normalization factor. After the transformations (7) and (8), the matrix element $(M'_l)_{22}$ turns to

$$(M'_l)_{22} = (\mathcal{M})_{2i} V_{i2} = N_2 (H'_l)_{21}, \quad (12)$$

in terms of (10) and (11). The remaining condition for the NNI form is realized in $(H'_l)_{21} = 0$. For the purpose, we impose U to satisfy the following condition,

$$(H'_l)_{21} = U_{i2}^* (H_l)_{ij} U_{j1} = 0. \quad (13)$$

Next we transform the neutrino mass matrix M_ν into the simple form (3). This is done by choosing U in (5) as

$$(M'_\nu)_{21} = (M'_\nu)_{12} = U_{i1} (M_\nu)_{ij} U_{j2} = 0, \quad (14)$$

$$(M'_\nu)_{31} = (M'_\nu)_{13} = U_{i1} (M_\nu)_{ij} U_{j3} = 0. \quad (15)$$

These (14) and (15) are contended if U is satisfied in the following equation

$$U_{i1}^* (M_\nu)_{ij}^* = N_4 U_{j1}, \quad (16)$$

where N_4 is an normalization factor. The solution U_{i1} in this equation is obtained by $U = (U_{i1}, U_{i2}, U_{i3})$ which diagonalize $M_\nu^\dagger M_\nu$.

Once U_{i1} is obtained, U_{i2} is successively determined by the condition (13), and U_{i3} is also determined by the orthogonality conditions with the other two vectors U_{i1} and U_{i2} . The explicit forms of U_{i2} and U_{i3} are written with U_{i1} as

$$U_{i2} = N_5 \epsilon_{ijk} U_{j1}^* (H_l^* U^*)_{k1}, \quad (17)$$

$$U_{i3} = N_6 ((H_l)_{ij} U_{j1} - (H'_l)_{i1} U_{i1}), \quad (18)$$

where N_5 and N_6 are normalization factors.

Finally we show that the number of parameters are reduced to 12 by the ambiguity of the six phases containing in the normalization factors of eigenstates U_{i1}, U_{i2}, U_{i3} and V_{i1}, V_{i2}, V_{i3} . The redefinitions $U \rightarrow U P^\dagger(\delta_\nu)$ and $V \rightarrow V P^\dagger(\delta_R)$ do not change the forms of lepton mass matrices (3) and (4), where $P(\delta_\nu)$ and $P(\delta_R)$ are diagonal phase matrices. By these redefinitions, (5) and (6) become

$$M''_\nu = P^\dagger(\delta_\nu) U^T M_\nu U P^\dagger(\delta_\nu) = P^\dagger(\delta_\nu) M'_\nu P^\dagger(\delta_\nu), \quad (19)$$

$$M''_l = P(\delta_\nu) U^\dagger M_l V P^\dagger(\delta_R) = P(\delta_\nu) M'_l P^\dagger(\delta_R). \quad (20)$$

We can choose $P(\delta_\nu)$ so as to absorb three phases of M'_ν and as a result one phase degree of freedom is only left in M''_ν . Introducing another phase matrix $P(\delta_L)$ to separate the phase factor from M'_l , we can rewrite (20) as

$$M''_l = P(\delta_\nu)P^\dagger(\delta_L) \times P(\delta_L)U^\dagger M_l V P^\dagger(\delta_R) \quad (21)$$

$$= P(\delta_\nu)P^\dagger(\delta_L) \times P(\delta_L)M'_l P^\dagger(\delta_R) \quad (22)$$

$$= P(\delta)\hat{M}_l, \quad (23)$$

where $P(\delta) \equiv P(\delta_\nu)P^\dagger(\delta_L)$ is a diagonal phase matrix and $\hat{M} \equiv P(\delta_L)U^\dagger M_l V P^\dagger(\delta_R)$ is a real symmetric matrix of NNI form [7, 10]. $P(\delta_L)$ and $P(\delta_R)$ can be chosen so that five phases of M'_l are absorbed.

By replacement M''_ν with M_ν and M''_l with M_l , we can always choose the lepton mass matrices as

$$M_\nu = m_{\nu_3} \begin{pmatrix} a_\nu & 0 & 0 \\ 0 & b_\nu & d_\nu \\ 0 & d_\nu & c_\nu \end{pmatrix}, \quad (24)$$

$$M_l = P(\delta)\hat{M}_l \quad (25)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i\theta_2) & 0 \\ 0 & 0 & \exp(i\theta_3) \end{pmatrix} \times m_3 \begin{pmatrix} 0 & a & 0 \\ c & 0 & b \\ 0 & d & e \end{pmatrix}, \quad (26)$$

where m_{ν_3} and m_3 are respectively the heaviest neutrino mass and the heaviest charged lepton mass and M_ν is complex symmetric including only one phase degree of freedom implicitly in some components. In (26) we make $(P(\delta))_{11} = 1$ using the remaining freedom in $P(\delta_L)$. Note that M_ν and M_l can be chosen as (24) and (26) at any scale. As mentioned in the introduction, the above representation (24) of M_ν only contributes to the $\nu_\mu - \nu_\tau$ mixing. The lepton mass matrices (24) and (26) have twelve parameters including three phases and these parameters are just the same as the number of physical quantities.

3 Physical Quantities and Twelve Parameters

In this section, our purpose is to study the relations between the physical quantities and twelve parameters for the charged lepton mass matrix M_l of (24) and the neutrino mass matrix M_ν of (26). The mass matrices (24) and (26) should generate correct mass eigenvalues. Introducing the three charged lepton mass eigenvalues for \hat{M}_l in (25) as input parameters, M_l , which has five free parameters at first, is parametrized by two free parameters. In the same way, M_ν is also parametrized by two free parameters. Then, the remaining six free parameters (each of \hat{M}_l , M_ν and $P(\delta)$ has two parameters) can be determined by the MNS matrix denoted by V_{MNS} , which should be fixed by the physical quantities,

$$V_{MNS}^\dagger = U_\nu^\dagger P(\delta) O_l, \quad (27)$$

where U_ν is the unitary matrix which diagonalize M_ν and O_l is the orthogonal matrix which diagonalize $\hat{M}_l \hat{M}_l^T$.

At first, let us consider the charged lepton mass matrix M_l . We apply the work for the quark sector done by Harayama and Okamura [11] to the lepton sector. $\hat{M}_l \hat{M}_l^T$ is a real symmetric matrix which can be diagonalized by orthogonal matrix O_l as $O_l^T \hat{M}_l \hat{M}_l^T O_l = m_3^2 \text{diag}(\xi_1^2, \xi_2^2, \xi_3^2)$, where $\xi_1 \equiv m_1/m_3$, $\xi_2 \equiv m_2/m_3$, $\xi_3 \equiv m_3/m_3 = 1$ and $m_i (i = 1, 2, 3)$ are the charged lepton masses.

As the results, we can parametrize the charged lepton mass matrix as

$$\hat{M}_l = m_3 \begin{pmatrix} 0 & qz/y & 0 \\ q/(yz) & 0 & b \\ 0 & d & y^2 \end{pmatrix}, \quad (28)$$

where the matrix elements b and d are expressed as

$$b = \sqrt{\frac{p+1-y^4 \pm R}{2} - \frac{q^2}{y^2 z^2}}, \quad (29)$$

$$d = \sqrt{\frac{p+1-y^4 \mp R}{2} - \frac{q^2 z^2}{y^2}}, \quad (30)$$

and

$$R = \sqrt{(1+p-y^4)^2 - 4(p+q^4) + 4q^2 y^2 \left(z^2 + \frac{1}{z^2}\right)}, \quad (31)$$

$$q^2 = \xi_1 \xi_2, \quad (32)$$

$$p = \xi_1^2 + \xi_2^2. \quad (33)$$

Thus, M_l has only two independent free parameters y and z .

The orthogonal matrix, O_l , which diagonalize $\hat{M}_l \hat{M}_l^T$ is also represented using only two independent free parameters as follows;

$$O_l = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} = \frac{1}{f_i} \begin{pmatrix} (\xi_i^2 - b^2 - c^2)ad \\ (\xi_i^2 - a^2)be \\ (\xi_i^2 - a^2)(\xi_i^2 - b^2 - c^2) \end{pmatrix}, \quad (34)$$

where $a = qz/y$, $c = q/(yz)$ and f_i is normalization factor.

Next we consider the neutrino mass matrix M_ν . $M_\nu^\dagger M_\nu$ should generate correct mass eigenvalues as $U_\nu^\dagger M_\nu^\dagger M_\nu U_\nu = D_\nu^2 = m_{\nu_3}^2 \text{diag}(\xi_{\nu_1}^2, \xi_{\nu_2}^2, \xi_{\nu_3}^2)$ by the unitary matrix U_ν , where $\xi_{\nu_1} \equiv m_{\nu_1}/m_{\nu_3}$, $\xi_{\nu_2} \equiv m_{\nu_2}/m_{\nu_3}$, $\xi_{\nu_3} \equiv m_{\nu_3}/m_{\nu_3} = 1$ and $m_{\nu_i} (i = 1, 2, 3)$ are the neutrino masses. In general the matrix $M_\nu^\dagger M_\nu$ can be parametrized without loss of generality as follows;

$$M_\nu^\dagger M_\nu = m_{\nu_3}^2 \begin{pmatrix} \xi_{\nu_1}^2 & 0 & 0 \\ 0 & 1 - C + \xi_{\nu_2}^2 & -\sqrt{(1-C)(C - \xi_{\nu_2}^2)} e^{-i\theta_1} \\ 0 & -\sqrt{(1-C)(C - \xi_{\nu_2}^2)} e^{i\theta_1} & C \end{pmatrix}, \quad (35)$$

using two parameters C and θ_1 so that trace and determinant are invariant under unitary transformations. The above matrix can be diagonalized by

$$U_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\theta_1} \cos \phi & -\sin \phi \\ 0 & \sin \phi & e^{i\theta_1} \cos \phi \end{pmatrix}, \quad (36)$$

where

$$\cos \phi = \sqrt{\frac{C - \xi_{\nu_2}^2}{1 - \xi_{\nu_2}^2}}, \quad \sin \phi = \sqrt{\frac{1 - C}{1 - \xi_{\nu_2}^2}}. \quad (37)$$

The neutrino mass matrix M_ν reconstructed from U_ν and D_ν is

$$\begin{aligned} M_\nu &= U_\nu^* D_\nu U_\nu^\dagger \\ &= m_{\nu_3} \begin{pmatrix} \xi_{\nu_1} & 0 & 0 \\ 0 & A - B \cos 2\phi & -e^{-i\theta_1} B \sin 2\phi \\ 0 & -e^{-i\theta_1} B \sin 2\phi & e^{-2i\theta_1} (A + B \cos 2\phi) \end{pmatrix}, \end{aligned} \quad (38)$$

where

$$A = \frac{1}{2}(1 + \xi_{\nu_2} e^{2i\theta_1}), \quad B = \frac{1}{2}(1 - \xi_{\nu_2} e^{2i\theta_1}). \quad (39)$$

(28) and (38) are the exact parametrization of the lepton mass matrices. These simple form can be obtained without any approximations.

Finally we obtain the MNS matrix by using (26), (34) and (36) as follows;

$$V_{MNS}^\dagger = U_\nu^\dagger P(\delta) O_l = \begin{pmatrix} \alpha_i \\ \beta_i \cos \phi e^{i(\theta_1 + \theta_2)} + \gamma_i \sin \phi e^{i\theta_3} \\ -\beta_i \sin \phi e^{i\theta_2} + \gamma_i \cos \phi e^{i(\theta_3 - \theta_1)} \end{pmatrix}, \quad (40)$$

where α_i , β_i and γ_i are expressed by y and z in (34).

In (40), six independent parameters, $y, z, \phi, \theta_1, \theta_2, \theta_3$, are included. y, z, ϕ correspond to three mixing angles and $\theta_1, \theta_2, \theta_3$ correspond to three CP phases. If the contribution of the charged lepton mass matrix to $\nu_\mu - \nu_\tau$ mixing is small, we can identify ϕ with the large $\nu_\mu - \nu_\tau$ mixing angle.

4 The MNS Matrix

In this section, we investigate the structure of the MNS matrix under the approximation that the mass ratios of the charged leptons $\xi_1 (= m_1/m_3)$ and $\xi_2 (= m_2/m_3)$ are small quantities compared with $\xi_3 = 1 (= m_3/m_3)$. In addition, we adopt the assumptions $y \sim O(1), z \sim O(1)$ in the charged lepton mass matrix M_l , based on the expectation that it has the same structure as the quark mass matrices, as mentioned in the introduction. We roughly estimate the values of y and z from the experimental values and check the validity of the above

assumptions. Then, we also study the structure of neutrino mass matrix M_ν under the condition that the maximal $\nu_\mu - \nu_\tau$ mixing is derived from V_{MNS} , in each case that neutrinos have hierarchical masses ($\xi_{\nu_1} \ll \xi_{\nu_2} \ll 1$) or degenerate masses ($\xi_{\nu_1} \sim \xi_{\nu_2} \sim 1$).

Let us start from the exact form of (28). The mass hierarchy of the charged leptons, $\xi_1 \ll \xi_2 \ll 1$, and the assumptions, $y \sim O(1)$ and $z \sim O(1)$, lead to the results,

$$R \sim (1 - y^4) - \frac{1 + y^4}{1 - y^4} p, \quad (41)$$

and then substituting (41) for R in (29) and (30), we obtain either case I or case II, which is corresponding to the sign in front of R in (29) and (30),

$$\text{case I : } b \sim \sqrt{1 - y^4} \sim O(1), \quad d \sim \sqrt{\frac{p}{1 - y^4}} \ll O(1), \quad (42)$$

$$\text{case II : } b \sim \sqrt{\frac{p}{1 - y^4}} \ll O(1), \quad d \sim \sqrt{1 - y^4} \sim O(1). \quad (43)$$

The case I and II have quite different structures at the following point. In the case I, d is a small quantity compared with b , oppositely in the case II, b is a small quantity compared with d . This leads to a different consequence for the mixing angle. Actually it is shown by Ref. [12] that the mass matrix leads to large mixing between the second and the third generations in the case I, in contrast it leads to small mixing in the case II. The case I is not consistent with our assumption that the charged lepton mass matrix has only small mixing. Therefore in the following discussion we only treat the case II and do not describe a detail calculation for the case I.

Next we adopt the assumption (43) in the MNS matrix V_{MNS}^\dagger of (40) and we obtain

$$V_{MNS}^\dagger \simeq \begin{pmatrix} 1 & qyz/\sqrt{p} & -qz\sqrt{1 - y^4}/y \\ -qyz \cos \phi e^{i(\theta_1 + \theta_2)}/\sqrt{p} & \cos \phi e^{i(\theta_1 + \theta_2)} & \sin \phi e^{i\theta_3} \\ qyz \sin \phi e^{i\theta_2}/\sqrt{p} & -\sin \phi e^{i\theta_2} & \cos \phi e^{i(\theta_3 - \theta_1)} \end{pmatrix}, \quad (44)$$

at leading order approximation. Here we roughly estimate the values of y and z from the experimental values and check the validity of the assumptions $y \sim O(1)$ and $z \sim O(1)$. Introducing the experimental values,

$$m_e(M_Z) \sim 0.4867 \text{ MeV}, \quad (45)$$

$$m_\mu(M_Z) \sim 102.7 \text{ MeV}, \quad (46)$$

$$m_\tau(M_Z) \sim 1747 \text{ MeV}, \quad (47)$$

and $\phi = 45^\circ$ for the maximal $\nu_\mu - \nu_\tau$ mixing, (44) becomes the following;

$$V_{MNS}^\dagger \simeq \begin{pmatrix} 1 & 0.0689yz & -0.00405z\sqrt{1 - y^4}/y \\ -0.0487yz & 0.707 & 0.707 \\ 0.0487yz & -0.707 & 0.707 \end{pmatrix}, \quad (48)$$

up to phase factor of each matrix element. If y and z are fixed, we can determine the MNS matrix. As one of the examples, we obtain the MNS matrix in the case $y = 0.90$ and $z = 0.62$;

$$V_{\text{MNS}}^\dagger \simeq \begin{pmatrix} 1 & 0.038 & -0.002 \\ -0.027 & 0.707 & 0.707 \\ 0.027 & -0.707 & 0.707 \end{pmatrix}. \quad (49)$$

This result is consistent with the best fit values $\theta_{e\mu} \sim 2.2^\circ$ and $\theta_{e\tau} \sim 0^\circ$ [5, 13]. Therefore we recognize that the assumptions $y \sim O(1)$ and $z \sim O(1)$ are valid.

Finally we show the neutrino mass matrix M_ν in each case that the neutrinos have hierarchical masses or degenerate masses.

In the case that the neutrinos have hierarchical masses ($\xi_{\nu_1} \ll \xi_{\nu_2} \ll 1$),

$$M_\nu \simeq m_{\nu_3} \begin{pmatrix} \xi_{\nu_1} & 0 & 0 \\ 0 & 1/2 & -e^{-i\theta_1}/2 \\ 0 & -e^{-i\theta_1}/2 & e^{-2i\theta_1}/2 \end{pmatrix}, \quad (50)$$

and in the case that the neutrinos have degenerate masses ($\xi_{\nu_1} \sim \xi_{\nu_2} \sim 1$),

$$M_\nu \simeq m_{\nu_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta_1} \cos \theta_1 & i \sin \theta_1 \\ 0 & i \sin \theta_1 & e^{-i\theta_1} \cos \theta_1 \end{pmatrix}. \quad (51)$$

These simple forms (50) and (51) of neutrino mass matrix are almost same from weak scale to GUT scale, since the renormalization effects are small as it is possible to be neglected except for the special cases [14]. The smallness of the renormalization effects is based on the fact that each element of the neutrino mass matrix change into the logarithm of the energy.

If (50) and (51) are originated from some symmetry of fundamental theory and these forms are ensured by this symmetry, it will become important to explore such symmetry as future works.

5 Summary

In conclusion we have proposed exact parametrization of the lepton mass matrices under the assumption that the neutrinos are Majorana particles. We have chosen the form which has no contribution to $\nu_e - \nu_\mu$ mixing as the neutrino mass matrix M_ν and the NNI form as the charged lepton mass matrix M_l . This is the exact parametrization of the lepton mass matrices, which reflects the small $\nu_e - \nu_\mu$ mixing and the large $\nu_\mu - \nu_\tau$ mixing.

Let us comment about the differences between quark and lepton case in short. At first the number of parameters is the same as the number of the physical quantities in the lepton case, so there remains no redundant parameters unlike

the quark case [8]. Second, we cannot transform into NNI form both M_ν and M_l at the same time although we can do in the quark case. Because the number of the parameters included in the mass matrices is smaller than the number of the physical quantities in the lepton case.

Our simple parametrization may be useful to find some symmetry from the new physics beyond the standard model. As future works there remains the problem to find such symmetry.

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